MID TERM: TOPOLOGY BMATH2, WINTER 2026

Total time: 3 Hours. **All questions are compulsory**. Each question carries 6 marks. You may use results proved in class without proof, but you need to state the result clearly before using it. If a question specifically asks for the proof of a result covered in class, you must provide a detailed proof. If you wish to use a problem from a homework/assignment, supply its solution too. Without a proper explanation, only partial points/no points will be credited.

(1) Let A and B be disjoint compact sets of a Hausdorff space X. Show that there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

(2) Let X and Y be topological spaces. A map $f : X \to Y$ is called locally constant if for every $x \in X$, there exists an open set U_x containing x such that f is constant on U_x. If X is connected then show that a locally constant function f will be constant on X.

(3) Let (X, d) be a connected unbounded metric space. Show that for every $x \in X$, and $r > 0$ there exists $y \in X$ such that $d(x, y) = r$.

(4) Let A be a connected subset of $\mathbb{R}^n$ and $\epsilon > 0$. Show that $U(A, \epsilon) := \{x \in \mathbb{R}^n \mid d_A(x) < \epsilon\}$ is path connected. Here $d_A(\cdot)$ denotes the distance function from the set A, i.e. $d_A(x) := \inf_{a \in A} d(x, a),\ x \in \mathbb{R}^n$.

(5) Let $\mathbb{R}^\omega := \{(x_i)_{i=1}^\infty \mid x_i \in \mathbb{R}\}$ be the countable infinite product of $\mathbb{R}$. Let $\bar{d}$ be the standard bounded metric on $\mathbb{R}$, i.e., $\bar{d}(x, y) := \min\{|x - y|, 1\}$. Let us recall the definition of *uniform metric*

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) := \sup_i \bar{d}(x_i, y_i), \mathbf{x} = (x_i), \mathbf{y} = (y_i) \in \mathbb{R}^\omega$$

on $\mathbb{R}^\omega$. Denote the set

$$D(\mathbf{x}, \epsilon) := \prod_{i=1}^{\infty} (x_i - \epsilon, x_i + \epsilon), \quad 0 < \epsilon < 1, \mathbf{x} = (x_i) \in \mathbb{R}^\omega.$$

Show that $D(\mathbf{x}, \epsilon)$ is not an open set in the uniform metric topology $(\mathbb{R}^\omega, \bar{\rho})$. Moreover, let

$$\mathbb{R}^\star := \bigcup_{k \geq 2026} \{(x_i)_{i=1}^\infty \in \mathbb{R}^\omega \mid x_i = 0,\ \forall\ i \geq k\}.$$

Find the closure of $\mathbb{R}^\star$ in $(\mathbb{R}^\omega, \bar{\rho})$.